11.9.1 Exercises

For a link to all of the additional resources available for this section, click OSttS Chapter 11 materials.

In Exercises 1 - 20, use the pair of vectors $\vec{v}$ and $\vec{w}$ to find the following quantities.

- $\vec{v} \cdot \vec{w}$
- The angle θ (in degrees) between $\vec{v}$ and $\vec{w}$
- $\text{proj}_{\vec{w}}(\vec{v})$
- $\vec{q} = \vec{v} - \text{proj}_{\vec{w}}(\vec{v})$ (Show that $\vec{q} \cdot \vec{w} = 0$.)

In Exercises 1 - 20, find a polar representation for the complex number z and then identify $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, $\arg(z)$ and $\text{Arg}(z)$.

For help with these exercises, click one or more of the resources below:

- Finding the dot product of two vectors
- Finding the angle between two vectors

1. $\vec{v} = \langle -2, -7 \rangle$ and $\vec{w} = \langle 5, -9 \rangle$
2. $\vec{v} = \langle -6, -5 \rangle$ and $\vec{w} = \langle 10, -12 \rangle$
3. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$ and $\vec{w} = \left\langle 1, -\sqrt{3} \right\rangle$
4. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle -6, -8 \rangle$
5. $\vec{v} = \langle -2, 1 \rangle$ and $\vec{w} = \langle 3, 6 \rangle$
6. $\vec{v} = \left\langle -3\sqrt{3}, 3 \right\rangle$ and $\vec{w} = \left\langle -\sqrt{3}, -1 \right\rangle$
7. $\vec{v} = \langle 1, 17 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$
8. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 5, 12 \rangle$
9. $\vec{v} = \langle -4, -2 \rangle$ and $\vec{w} = \langle 1, -5 \rangle$
10. $\vec{v} = \langle -5, 6 \rangle$ and $\vec{w} = \langle 4, -7 \rangle$
11. $\vec{v} = \langle -8, 3 \rangle$ and $\vec{w} = \langle 2, 6 \rangle$
12. $\vec{v} = \langle 34, -91 \rangle$ and $\vec{w} = \langle 0, 1 \rangle$
13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$
18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$
19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$
21. A force of 1500 pounds is required to tow a trailer. Find the work done towing the trailer along a flat stretch of road 300 feet. Assume the force is applied in the direction of the motion.
22. Find the work done lifting a 10 pound book 3 feet straight up into the air. Assume the force of gravity is acting straight downwards.
23. Suppose Taylor fills her wagon with rocks and must exert a force of 13 pounds to pull her wagon across the yard. If she maintains a 15° angle between the handle of the wagon and the horizontal, compute how much work Taylor does pulling her wagon 25 feet. Round your answer to two decimal places.

24. In Exercise 61 in Section 11.8, two drunken college students have filled an empty beer keg with rocks which they drag down the street by pulling on two attached ropes. The stronger of the two students pulls with a force of 100 pounds on a rope which makes a 13° angle with the direction of motion. (In this case, the keg was being pulled due east and the student's heading was N77°E.) Find the work done by this student if the keg is dragged 42 feet.

25. Find the work done pushing a 200 pound barrel 10 feet up a 12.5° incline. Ignore all forces acting on the barrel except gravity, which acts downwards. Round your answer to two decimal places.

    HINT: Since you are working to overcome gravity only, the force being applied acts directly upwards. This means that the angle between the applied force in this case and the motion of the object is *not* the 12.5° of the incline!

26. Prove the distributive property of the dot product in Theorem 11.22.

27. Finish the proof of the scalar property of the dot product in Theorem 11.22.

28. Use the identity in Example 11.9.1 to prove the **Parallelogram Law**

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 = \frac{1}{2}\left[\|\vec{v}+\vec{w}\|^2 + \|\vec{v}-\vec{w}\|^2\right]$$

29. We know that $|x+y| \leq |x| + |y|$ for all real numbers x and y by the Triangle Inequality established in Exercise 36 in Section 2.2. We can now establish a Triangle Inequality for vectors. In this exercise, we prove that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

    (a) (Step 1) Show that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$.

    (b) (Step 2) Show that $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$. This is the celebrated Cauchy-Schwarz Inequality.[6] (Hint: To show this inequality, start with the fact that $|\vec{u}\cdot\vec{v}| = |\ \|\vec{u}\|\|\vec{v}\|\cos(\theta)\ |$ and use the fact that $|\cos(\theta)| \leq 1$ for all θ.)

    (c) (Step 3) Show that $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2|\vec{u}\cdot\vec{v}| + \|\vec{v}\|^2 \leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 = (\|\vec{u}\| + \|\vec{v}\|)^2$.

    (d) (Step 4) Use Step 3 to show that $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all pairs of vectors $\vec{u}$ and $\vec{v}$.

    (e) As an added bonus, we can now show that the Triangle Inequality $|z+w| \leq |z| + |w|$ holds for all complex numbers z and w as well. Identify the complex number $z = a+bi$ with the vector $u = \langle a, b\rangle$ and identify the complex number $w = c+di$ with the vector $v = \langle c, d\rangle$ and just follow your nose!

[6] It is also known by other names. Check out this site for details.

11.9.2 Answers

1. $\vec{v} = \langle -2, -7 \rangle$ and $\vec{w} = \langle 5, -9 \rangle$
   $\vec{v} \cdot \vec{w} = 53$
   $\theta = 45^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{5}{2}, -\frac{9}{2} \right\rangle$
   $\vec{q} = \left\langle -\frac{9}{2}, -\frac{5}{2} \right\rangle$

2. $\vec{v} = \langle -6, -5 \rangle$ and $\vec{w} = \langle 10, -12 \rangle$
   $\vec{v} \cdot \vec{w} = 0$
   $\theta = 90^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
   $\vec{q} = \langle -6, -5 \rangle$

3. $\vec{v} = \left\langle 1, \sqrt{3} \right\rangle$ and $\vec{w} = \left\langle 1, -\sqrt{3} \right\rangle$
   $\vec{v} \cdot \vec{w} = -2$
   $\theta = 120^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$
   $\vec{q} = \left\langle \frac{3}{2}, \frac{\sqrt{3}}{2} \right\rangle$

4. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle -6, -8 \rangle$
   $\vec{v} \cdot \vec{w} = -50$
   $\theta = 180^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 3, 4 \rangle$
   $\vec{q} = \langle 0, 0 \rangle$

5. $\vec{v} = \langle -2, 1 \rangle$ and $\vec{w} = \langle 3, 6 \rangle$
   $\vec{v} \cdot \vec{w} = 0$
   $\theta = 90^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0 \rangle$
   $\vec{q} = \langle -2, 1 \rangle$

6. $\vec{v} = \left\langle -3\sqrt{3}, 3 \right\rangle$ and $\vec{w} = \left\langle -\sqrt{3}, -1 \right\rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta = 60^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{3\sqrt{3}}{2}, -\frac{3}{2} \right\rangle$
   $\vec{q} = \left\langle -\frac{3\sqrt{3}}{2}, \frac{9}{2} \right\rangle$

7. $\vec{v} = \langle 1, 17 \rangle$ and $\vec{w} = \langle -1, 0 \rangle$
   $\vec{v} \cdot \vec{w} = -1$
   $\theta \approx 93.37^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \langle 1, 0 \rangle$
   $\vec{q} = \langle 0, 17 \rangle$

8. $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 5, 12 \rangle$
   $\vec{v} \cdot \vec{w} = 63$
   $\theta \approx 14.25^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{315}{169}, \frac{756}{169} \right\rangle$
   $\vec{q} = \left\langle \frac{192}{169}, -\frac{80}{169} \right\rangle$

9. $\vec{v} = \langle -4, -2 \rangle$ and $\vec{w} = \langle 1, -5 \rangle$
   $\vec{v} \cdot \vec{w} = 6$
   $\theta \approx 74.74^\circ$
   $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{3}{13}, -\frac{15}{13} \right\rangle$
   $\vec{q} = \left\langle -\frac{55}{13}, -\frac{11}{13} \right\rangle$

10. $\vec{v} = \langle -5, 6 \rangle$ and $\vec{w} = \langle 4, -7 \rangle$
    $\vec{v} \cdot \vec{w} = -62$
    $\theta \approx 169.94^\circ$
    $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{248}{65}, \frac{434}{65} \right\rangle$
    $\vec{q} = \left\langle -\frac{77}{65}, -\frac{44}{65} \right\rangle$

11. $\vec{v} = \langle -8, 3\rangle$ and $\vec{w} = \langle 2, 6\rangle$
 $\vec{v}\cdot\vec{w} = 2$
 $\theta \approx 87.88^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1}{10}, \frac{3}{10}\right\rangle$
 $\vec{q} = \left\langle -\frac{81}{10}, \frac{27}{10}\right\rangle$

12. $\vec{v} = \langle 34, -91\rangle$ and $\vec{w} = \langle 0, 1\rangle$
 $\vec{v}\cdot\vec{w} = -91$
 $\theta \approx 159.51^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -91\rangle$
 $\vec{q} = \langle 34, 0\rangle$

13. $\vec{v} = 3\hat{\imath} - \hat{\jmath}$ and $\vec{w} = 4\hat{\jmath}$
 $\vec{v}\cdot\vec{w} = -4$
 $\theta \approx 108.43^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, -1\rangle$
 $\vec{q} = \langle 3, 0\rangle$

14. $\vec{v} = -24\hat{\imath} + 7\hat{\jmath}$ and $\vec{w} = 2\hat{\imath}$
 $\vec{v}\cdot\vec{w} = -48$
 $\theta \approx 163.74^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \langle -24, 0\rangle$
 $\vec{q} = \langle 0, 7\rangle$

15. $\vec{v} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath}$ and $\vec{w} = \hat{\imath} - \hat{\jmath}$
 $\vec{v}\cdot\vec{w} = 0$
 $\theta = 90^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \langle 0, 0\rangle$
 $\vec{q} = \left\langle \frac{3}{2}, \frac{3}{2}\right\rangle$

16. $\vec{v} = 5\hat{\imath} + 12\hat{\jmath}$ and $\vec{w} = -3\hat{\imath} + 4\hat{\jmath}$
 $\vec{v}\cdot\vec{w} = 33$
 $\theta \approx 59.49^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle -\frac{99}{25}, \frac{132}{25}\right\rangle$
 $\vec{q} = \left\langle \frac{224}{25}, \frac{168}{25}\right\rangle$

17. $\vec{v} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$
 $\vec{v}\cdot\vec{w} = \frac{\sqrt{6}-\sqrt{2}}{4}$
 $\theta = 75^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{1-\sqrt{3}}{4}, \frac{\sqrt{3}-1}{4}\right\rangle$
 $\vec{q} = \left\langle \frac{1+\sqrt{3}}{4}, \frac{1+\sqrt{3}}{4}\right\rangle$

18. $\vec{v} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\rangle$ and $\vec{w} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$
 $\vec{v}\cdot\vec{w} = \frac{\sqrt{2}-\sqrt{6}}{4}$
 $\theta = 105^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{2}-\sqrt{6}}{8}, \frac{3\sqrt{2}-\sqrt{6}}{8}\right\rangle$
 $\vec{q} = \left\langle \frac{3\sqrt{2}+\sqrt{6}}{8}, \frac{\sqrt{2}+\sqrt{6}}{8}\right\rangle$

19. $\vec{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$ and $\vec{w} = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$
 $\vec{v}\cdot\vec{w} = -\frac{\sqrt{6}+\sqrt{2}}{4}$
 $\theta = 165^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, \frac{\sqrt{3}+1}{4}\right\rangle$
 $\vec{q} = \left\langle \frac{\sqrt{3}-1}{4}, \frac{1-\sqrt{3}}{4}\right\rangle$

20. $\vec{v} = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle$ and $\vec{w} = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right\rangle$
 $\vec{v}\cdot\vec{w} = \frac{\sqrt{6}+\sqrt{2}}{4}$
 $\theta = 15^\circ$
 $\text{proj}_{\vec{w}}(\vec{v}) = \left\langle \frac{\sqrt{3}+1}{4}, -\frac{\sqrt{3}+1}{4}\right\rangle$
 $\vec{q} = \left\langle \frac{1-\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4}\right\rangle$

21. $(1500\,\text{pounds})(300\,\text{feet})\cos(0^\circ) = 450,000$ foot-pounds

22. $(10\,\text{pounds})(3\,\text{feet})\cos(0^\circ) = 30$ foot-pounds

23. $(13\,\text{pounds})(25\,\text{feet})\cos(15^\circ) \approx 313.92$ foot-pounds

24. $(100\,\text{pounds})(42\,\text{feet})\cos(13^\circ) \approx 4092.35$ foot-pounds

25. $(200\,\text{pounds})(10\,\text{feet})\cos(77.5^\circ) \approx 432.88$ foot-pounds